On Zero Controllability of Evolution Equations

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June 17, 2009

Abstract

The exact controllability to the origin for linear evolution control equation is considered.The problem is investigated by its transformation to infinite linear moment problem.

Conditions for the existence of solution for infinite linear moment problem has been obtained. The obtained results are applied to the zero controllability for control evolution equations.

Introduction

Let X be a separable complex Hilbert space.

Given sequences $\{c_n, n = 1, 2 \ldots,\}$ and $\{x_n \in X, n = 1, 2 \ldots,\}$ find necessary and sufficient conditions for the existence of an element $g \in X$ such that

$$c_n = (x_n, g), n = 1, 2 \ldots, .$$

The problem formulated above is called the linear moment problem. It has a long history and many applications in geometry, physics, mechanics.

The goal of this paper is to establish necessary and sufficient conditions of exact null-controllability for linear evolution control equations with unbounded input operator by transformation of exact null-controllability problem (controllability to the origin) to linear infinite moment problem.

It is well-known, that if the sequence $\{x_n, n = 1, 2, ...,\}$forms a Riesz basic in the closure of its linear span, the linear moment problem has a solution if and only if $\sum_{n=1}^{\infty} |c_n|^2 < \infty$ and vice-versa [3], [7], [16], [17]. This well-known fact is one of main tools for the controllability analysis of various partial hyperbolic control equations and functional differential control systems of neutral type.

However the sequence $\{x_n, n = 1, 2, ..., \}$ doesn't need to be a Riesz basic for the solvability of linear moment problem. This case appears under the investigation of the controllability of parabolic control equations or hereditary functional differential control systems. In this paper we consider the zero controllability of control evolution equations for the case when the sequence $\{x_n, n = 1, 2, ..., \}$ of the moment problem obtained by the transformation of the source control problem doesn't form a Riesz basic in its closed linear span.

1 Problem statement

Let X, U be complex Hilbert spaces, and let A be infinitesimal generator of strongly continuous C_0-semigroups $S(t)$ in X [8],[10]. Consider the abstract evolution control equation [8], [10]

$$\dot{x}(t) = Ax(t) + Bu(t), x(0) = x_0, \ 0 \le t < +\infty, \tag{1.1}$$

where $x(t), \ x_0 \in X, u(t), u_0 \in U, \ B : U \to X$ is a linear possibly unbounded operator, $W \subset X \subset V$ are Hilbert spaces with continuous dense injections, where $W = D(A)$ equipped with graphic norm, $V = W^*$, the operator B is a bounded operator from U to V (see more details in [14], [4], [11], [15]).

It is well-known that [4], [11],[14], [15]), etc. :

• for each $t \ge 0$ the operator $S(t)$ has an unique continuous extension $\mathcal{S}(t)$ on the space V and the family of operators $\mathcal{S}(t) : V \to V$ is the semigroup in the class C_0 with respect to the norm of V and the corresponding infinitesimal generator $\mathcal{A}$ of the semigroup $\mathcal{S}(t)$ is the closed dense extension of the operator A on the space V with domain $D(\mathcal{A}) = X$;

• the sets of eigenvalues and of generalized eigenvectors of operators $\mathcal{A}, \mathcal{A}^*$ and A, A^* are the same;

• for each $\mu \notin \sigma(A)$ the resolvent operator $R_A(\mu)$ has a unique continuous extension to the resolvent operator $\mathcal{R}_A(\mu) : V \to X$;

• a mild solution $x(t, x_0, u(\cdot))$ of equation (1.1) with initial condition $x(0) = x_0$ is obtained by the following representation formula

$$x(t, x_0, u(\cdot)) = S(t)x_0 + \int_0^t \mathcal{S}(t-\tau)Bu(\tau)d\tau, \tag{1.2}$$

where the integral in (2.3) is understood in the Bochner's sense [8]. To assure $x(t, x_0, u(\cdot)) \in X, \ \forall x_0 \in X, u(\cdot) \in L_2^{\text{loc}}[0, +\infty), t \ge 0,$ we assume

that $\int\limits_0^t \mathcal{S}(t-\tau)Bu(\tau)d\tau \in X$ for any $u(\cdot) \in L_2^{\mathrm{loc}}[0,+\infty), t \geq 0$ [14], [15].

Definition 1.1 *Equation (1.1) is said to be exact null-controllable on* $[0,t_1]$ *by controls vanishing after time moment* t_2 *if for each* $x_0 \in X$ *there exists a control* $u(\cdot) \in L_2([0,t_2], U), u(t) = 0$ *a.e. on* $[t_2,+\infty)$ *such that*

$$x(t_1, x_0, u(\cdot)) = 0. \tag{1.3}$$

1.1 The assumptions

The assumptions on A are listed below.

1. The operators A has purely point spectrum σ with no finite limit points. Eigenvalues of A have finite multiplicities.

2. There exists $T \geq 0$ such that all mild solutions of the equation $\dot{x}(t) = Ax(t)$ are expanded in a series of generalized eigenvectors of the operator A converging uniformly for any $t \in [T_1, T_2], T < T_1 < T_2$.

2 Main results

2.1 One input case

For the sake of simplicity we consider the following:

1. The operator A has all the eigenvalues with multiplicity 1.

2. $U = \mathbb{R}$ (one input case). It means that the possibly unbounded operator $B : U \to \mathbb{R}$ is defined by an element $b \in V$, i.e. equation (1.1) can be written in the form

$$\dot{x}(t) = Ax(t) + bu(t), x(0) = x_0, b \in V, \ 0 \leq t < +\infty. \tag{2.1}$$

   The operator defined by $b \in V$ is bounded if and only if $b \in X$.

Let the eigenvalues $\lambda_j \in \sigma, j = 1, 2, \ldots$ of the operator A be enumerated in the order of non-decreasing of their absolute values, and let $\varphi_j, \psi_j, j = 1, 2, \ldots$,be eigenvectors of the operator A and the adjoint operator A^* respectively. It is well-known, that

$$(\varphi_k, \psi_j) = \delta_{kj}, \ j,k = 1, 2 \ldots, \tag{2.2}$$

where δ_{kj}, $j,k=1,2\ldots$is the Kroneker delta.

Denote:

$$x_j(t)=\left(x(t,x_0,u(\cdot)),\psi_j\right),\ x_{0j}=\left(x_0,\psi_j\right),\ b_j=\left(b,\psi_j\right),\ j=1,2,.... \tag{2.3}$$

All scalar products in (2.3) are correctly defined, because $\psi_j\in W$, $b\in V=W^*$.

Theorem 2.1 *For equation (1.1) to be exact null-controllable on* $[0,t_1]$, $t_1>T$, *by controls vanishing after time moment* t_1-T, *it is necessary and sufficient that the following infinite moment problem*

$$x_{0j}=-\int_0^{t_1-T}e^{-\lambda_j\tau}b_ju(\tau)\,d\tau,\ j=1,2,... \tag{2.4}$$

with respect to $u(\cdot)\in L_2[0,t_1-T]$ *is solvable for any* $x_0\in X$.

Proof. Necessity. Multiplying (1.1) by ψ_j, $j=1,2,...,$and using (2.3) we obtain

$$\begin{aligned}\dot{x}_j(t) &= \left(Ax(t),\psi_j\right)+b_ju(t)=\left(x(t),A^*\psi_j\right)+b_ju(t)= \\ &= \lambda_jx_j(t)+b_ju(t),j=1,2,...,.\end{aligned} \tag{2.5}$$

Here $x_j(t),\dot{x}_j(t)$ and $b_j, j=1,2,...,$ are well-defined because $\psi_j\in W,\ \dot{x}(t),Ax(t),b\in V=W^*$.

From (2.5) it follows that

$$x_j(t)=e^{\lambda_jt}\left(x_{j0}+\int_0^te^{-\lambda_jt}b_ju(\tau)\,d\tau\right),j=1,2,...,. \tag{2.6}$$

In accordance with the definition of exact null-controllability there exists $u(\cdot)\in L_2([0,t_1-T],U),u(t)=0$ a.e. on $[t_1-T,+\infty)$ such that (1.3) holds. Using $u(t)$ and t_1 in (2.6), we obtain by (1.3) and (2.5), that

$$x_j(t_1)=e^{\lambda_jt_1}\left(x_{j0}+\int_0^{t_1-T}e^{-\lambda_jt}b_ju(\tau)\,d\tau\right)=0,j=1,2,...,. \tag{2.7}$$

Hence we have (2.4) to be true. This proves the necessity.

Sufficiency. Let the control $u(\cdot)\in L_2([0,t_1-T],U),u(t)=0$ a.e. on $[t_1-T,+\infty)$ satisfies (2.4). It follows from (2.4) and (2.7) that

$$x_j(t_1-T)=\left(x(t_1-T),\psi_j\right)=0,j=1,2,.... \tag{2.8}$$

Denote $z(t) = x(t + t_1 - T)$, $t \geq T$.Obviously, $z(t)$ is a mild solution of the equation $\dot{z}(t) = Az(t)$ with initial condition $z(0) = x(t_1 - T)$ .By assumption 3 (see the list of assumptions) $z(t)$ is expanded in a series

$$z(t) = \sum_{j=1}^{\infty} e^{\lambda_j t} \left(x(t_1 - T), \psi_j \right), t \geq T, \tag{2.9}$$

so by (2.8) and (2.9) we obtain
$z(t) = x(t + t_1 - T, x_0, u(\cdot)) \equiv 0, t \geq T \Leftrightarrow x(t, x_0, u(\cdot)) \equiv 0, t \geq t_1.$
This proves the sufficiency.

2.2 Solution of moment problem (2.4)

The solvability of moment problem (2.4) for each $x_0 \in X$ essentially depends on the properties of eigenvalues λ_j, $j = 1, 2, ...,$.

If the sequence of exponents $\left\{ e^{-\lambda_n t} b_n, n = 1, 2, ..., \right\}$forms a Riesz basic in $L_2[0, t_1 - T]$,then the moment problem

$$c_j = -\int_0^{t_1 - T} e^{-\lambda_j \tau} b_j u(\tau) \, d\tau, \; j = 1, 2, ... \tag{2.10}$$

is solvable if and only if

$$\sum_{j=1}^{\infty} |c_j|^2 < \infty \tag{2.11}$$

There are very large number of papers and books devoted to conditions for sequence of exponents to be a Riesz basic. All these conditions can be used for sufficient conditions of zero controllability of equation (1.1). They are very useful for the investigation of the zero controllability of hyperbolic partial control equations and functional differential control systems of neutral type [13].

However moment problem (2.10) may also be solvable when the sequence $\left\{ e^{-\lambda_n t} b_n, n = 1, 2, ..., \right\}$ doesn't form a Riesz basic in $L_2[0, t_1 - T]$. Below we will try to find more extended controllability conditions which are applicable for the case when the sequence $\left\{ e^{-\lambda_n t} b_n, n = 1, 2, ..., \right\}$ doesn't form a Riesz basic in $L_2[0, t_1 - T]$.

Definition 2.1 *The sequence $\{x_j \in X, j = 1, 2, ..., \}$ is said to be minimal, if there no element of the sequence belonging to the closure of the linear span of others. By other words,*

$$x_j \notin \overline{\text{span}} \{x_k \in X, k = 1, 2, ..., k \neq j\}.$$

The investigation of the controllability problem defined above is based on the following result of Boas [2] (see also [3] and [18]).

Theorem *Let $x_j \in X, j = 1, 2, ...,$. The linear moment problem*

$$c_j = (x_j, g), j = 1, 2, ...$$

has a solution $g \in X$ for each square summable sequence $\{c_j, j = 1, 2, ...\}$ if and only if there exists a positive constant γ such that all the inequalities

$$\gamma \sum_{k=1}^{n} |c_k|^2 \leq \left\| \sum_{j=1}^{n} c_j x_j \right\|^2, n = 1, 2, ...,. \tag{2.12}$$

are valid.

Let $\{x_j \in X, j = 1, 2, ...,\}$ a sequence of elements of X , and let

$$G_n = \{(x_i, x_j), i, j = 1, 2, ..., n\}$$

be the Gram matrix of n first elements $\{x_1, ..., x_n\}$ of above sequence. Denote by $\gamma_n^{\min}$ the minimal eigenvalue of the $n \times n$-matrix G_n.Each minimal sequence $\{x_j \in X, j = 1, 2, ...,\}$ is linear independent, hence any first n elements $\{x_1, ..., x_n\}$, $n = 1, 2, ...,$ of this sequence are linear independent, so $\gamma_n^{\min} > 0$, $\forall n = 1, 2, ...,$. It is easily to show that the sequence $\left\{\gamma_n^{\min}, n = 1, 2, ...,\right\}$ decreases , so there exists $\lim\limits_{n\to\infty} \gamma_n^{\min} \geq 0$.

Definition 2.2 *The sequence $\{x_j \in X, j = 1, 2, ...,\}$ is said to be strongly minimal, if $\gamma^{\min} = \lim\limits_{n\to\infty} \gamma_n^{\min} > 0$.*

It is well-known that for Hermitian $n \times n$-matrix $G_n = \{(x_j, x_k), \ j, k = 1, 2, ..., n\}$

$$\gamma_n^{\min} \sum_{k=1}^{n} |c_k|^2 \leq \sum_{j=1}^{n} \sum_{k=1}^{n} c_j (x_j, x_k) \overline{c_k}, n = 1, 2, ...,. \tag{2.13}$$

From the well-known formula $\sum_{j=1}^{m} \sum_{k=1}^{m} c_j (x_j, x_k) \overline{c_k} = \left\| \sum_{j=1}^{m} c_j x_j \right\|^2$,(2.12) and the inequality $\gamma_n^{\min} \geq \gamma^{\min} > 0$ it follows that

$$\gamma^{\min} \sum_{k=1}^{n} |c_k|^2 \leq \left\| \sum_{j=1}^{n} c_j x_j \right\|^2 \tag{2.14}$$

Hence the above theorem can be reformulated as follows

Theorem 2.2 *The linear moment problem*

$$c_j = (x_j, g), j = 1, 2, ... \tag{2.15}$$

has a solution $g \in X$ for any sequence $\{c_n, n = 1, 2, ...\}$, $\sum\limits_{j=1}^{\infty} c_j^2 < \infty$ if and only if the sequence $\{x_n, n = 1, 2, .., \}$ is strongly minimal.

3 Solution of the exact null-controllability problem.

Theorem 3.1 *For equation (1.1) to be exact null-controllable on $[0, t_1]$, $t_1 > T$, by controls vanishing after time moment $t_1 - T$, it is necessary, that the sequence*

$$\left\{e^{-\lambda_j \tau} b_j, t \in [0, t_1 - T], \ j = 1, 2, ..., \right\} \tag{3.1}$$

is minimal, and sufficient , that:

- *the sequence $\left\{e^{-\lambda_j \tau} b_j, t \in [0, t_1 - T], \ j = 1, 2, ...\right\}$ is strongly minimal;*
- $$\sum_{j=1}^{\infty} \left|\left(x_0, \psi_j\right)\right|^2 < +\infty, \forall x_0 \in X. \tag{3.2}$$

Proof. Necessity. If the problem (2.4) has a solution for any $x_0 \in X$,then it has a solution for any eigenvector $\varphi_k, k = 1, 2, ...$, of the operator A, so for each $k = 1, 2, ...$, there exists a function $u_k(\cdot) \in L_2[0, t_1 - T]$ such that

$$\left(\varphi_k, \psi_j\right) = -\int_0^{t_1 - T} e^{-\lambda_j \tau} b_j u_k(\tau) \, d\tau, \quad j = 1, 2, ..., . \tag{3.3}$$

The sequence $\{\varphi_k, k = 1, 2, ..., \}$ of eigenvectors of the operator A is biorthogonal to the sequence $\{\psi_k, k = 1, 2, ..., \}$ of eigenvectors of the operator A^*. Hence it follows from (3.3) and (2.2) that

$$\delta_{jk} = \left(\varphi_k, \psi_j\right) = -\int_0^{t_1 - T} e^{-\lambda_j \tau} b_j u_k(\tau) \, d\tau, j = 1, 2, ..., .$$

i.e. the sequence $\{-u_k(t), t \in [0, t_1 - T], \ k = 1, 2, ..., \}$ is biorthogonal to the sequence $\left\{e^{-\lambda_j t} b_j, t \in [0, t_1 - T], \ j = 1, 2, ..., \right\}$.

It proves the necessity.

Sufficiency. The sufficiency follows immediately from (3.2) and Theorem 2.2.

It proves the theorem.

3.1 The case of the strongly minimal sequence of eigenvectors of the operator A.

Obviously the sequence of eigenvectors of the operator A being considered is a minimal sequence.

Below we consider the operator A having the strongly minimal sequence of eigenvectors.

Theorem 3.2 *Let the sequence $\{\varphi_j, j = 1, 2, ...\}$ of eigenvectors of the operator A be strongly minimal.*

For equation (1.1) to be exact null-controllable on $[0, t_1]$, $t_1 > T$, by controls vanishing after time moment $t_1 - T$, it is necessary, that the sequence $\{e^{-\lambda_j \tau} b_j, t \in [0, t_1 - T],\ j = 1, 2, ...\}$ is minimal, and sufficient, that $\operatorname{Re}\lambda_j \geq \beta$ for some $\beta \in \mathbb{R}$ and the sequence $\{e^{-\lambda_j t} b_j, t \in [0, t_1 - T],\ j = 1, 2, ...\}$ is strongly minimal.

Proof. The necessity follows from Theorem 3.1.

Sufficiency. By Assumption 3 of the list of assumptions the series

$$\sum_{j=1}^{\infty} (x_0, \psi_j)\, e^{\lambda_j t} \varphi_j, \forall t > T \tag{3.4}$$

converges. Since the sequence $\{\varphi_j, j = 1, 2, ...\}$ of eigenvectors of the operator A is strongly minimal, then on account of property (2.10 there exists a number α such that

$$\begin{aligned} \alpha^2 \sum_{j=1}^{n} \left|(x_0, \psi_j)\right|^2 e^{2\operatorname{Re}\lambda_j t} &\leq \sum_{j=1}^{n} \sum_{k=1}^{n} (x_0, \psi_j)\, e^{\lambda_j t} (\varphi_j, \varphi_k)\, \overline{(x_0, \psi_k)} e^{\overline{\lambda_k} t}, \\ \forall x_0 &\in X,\ \forall n \in \mathbb{N},\ \forall t > T. \end{aligned} \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$\sum_{j=1}^{\infty} \left|(x_0, \psi_j)\right|^2 e^{2\operatorname{Re}\lambda_j t} < +\infty, \forall x_0 \in X, \forall t > T. \tag{3.6}$$

As $\operatorname{Re}\lambda_j \geq \beta$ for some $\beta \in \mathbb{R}$,we have by (3.6) that (3.2) holds.

In accordance with Theorem 3.1 condition (3.2) and the strong minimality of the sequence (3.1) imply the exact null-controllability of equation (1.1). It proves the theorem.

3.2 The case when the eigenvectors of the operator A form a Riesz basic

One of the important problems of the operator theory is the case when the generalized eigenvectors of the operator A being considered form a Riesz basic in X. The problem of expansion into a Riesz basic of eigenvectors of the operator A is widely investigated in the literature (see, for example, [1], [6], [7], [12] and references therein). Obviously the sequence of these vectors is strongly minimal. In this case one can set $T = 0$, so the Theorems 3.1, 3.2 and Lemma 3.1 can be proven with $T = 0$.

Theorem 3.3 *Let the sequence of operator A forms a Riesz basic in X.*

For equation (1.1) to be exact null-controllable on $[0, t_1], t_1 > T$, by controls vanishing after time moment $t_1 - T$, it is necessary and sufficient, that the sequence sequence$\{e^{-\lambda_j t} b_j, t \in [0, t_1 - T], \ j = 1, 2, ...\}$ is strongly minimal .

Proof. Let $\{c_j, j = 1, 2, ...,\}$ be any complex sequence satisfying the condition $\sum_{j=1}^{\infty} |c_j|^2 < \infty$.

Since the sequence $\{\varphi_j, j = 1, 2, ...,\}$ of eigenvectors of the operator A forms the Riesz basic, there exists a vector $x_0 \in X$ such that

$$c_j = (x_0, \psi_j), j = 1, 2, ...,$$

so in virtue of Theorem 2.1 the exact null controllability being considered in the paper is equivalent to the solvability of the linear moment problem

$$c_j = \int_0^{t_1 - T} e^{-\lambda_j \tau} b_j u(\tau) \, d\tau, \ j = 1, 2, ..., \tag{3.7}$$

for any complex sequence $\{c_j, j = 1, 2, ...,\}$ satisfying the condition $\sum_{j=1}^{\infty} |c_j|^2 < \infty$.

By above mentioned results of [2] and [3] the linear moment problem (3.7) is solvable for any complex sequence $\{c_j, j = 1, 2, ...,\}$ satisfying the condition $\sum_{j=1}^{\infty} |c_j|^2 < \infty$ if and only if the sequence $\{e^{-\lambda_j t} b_j, t \in [0, t_1 - T], \ j = 1, 2, ...\}$ is strongly minimal . It proves the theorem.

Obviously, the condition $b_j \neq 0, j = 1, 2, ...,$ is the necessary condition for the solvability of the moment problem (2.1).

Lemma 3.1 *If the sequence*

$$\left\{e^{-\lambda_j t}, t \in [0, t_1 - T], \ j = 1, 2, ...\right\} \tag{3.8}$$

is strongly minimal and

$$\inf_{n\in\mathbb{N}} |b_n| = \beta > 0 \tag{3.9}$$

holds, then the sequence $\left\{e^{-\lambda_j t}b_j, t \in [0, t_1 - T]\,,\ j = 1, 2, ...\right\}$ *is also strongly minimal.*

Proof. Let the sequence $\left\{e^{-\lambda_j t}, t \in [0, t_1 - T]\,,\ j = 1, 2, ...\right\}$ be strongly minimal. From (2.12) it follows that

$$\alpha \sum_{k=1}^{n} |c_k|^2 |b_j|^2 \leq \int_0^{t_1 - T} \left| \sum_{j=1}^{n} c_j e^{-\lambda_j t} b_j \right|^2 dt \tag{3.10}$$

for some positive α and for every finite sequence $\{c_1, c_2, ..., c_n\}$. By (3.9) and (3.10) we have

$$\gamma \sum_{k=1}^{n} |c_k|^2 \leq \int_0^{t_1 - T} \left| \sum_{j=1}^{n} c_j e^{-\lambda_j t} b_j \right|^2 dt, n = 1, 2, ..., \gamma = \alpha\beta > 0. \tag{3.11}$$

where $\gamma = \alpha\beta > 0$. It proves the lemma.

Example of strongly minimal sequence. Below we will prove that the sequence $\left\{e^{n^2\pi^2 t}, n = 1, 2, ..., t \in [0, t_1]\right\}$ is strongly minimal for any $t_1 > 0$.

Let $t_1 = 2t_2$. The series $\sum_{n=1}^{\infty} \frac{1}{n^2\pi^2}$ converges and $(n+1)^2 - n^2 \geq 1$, so the sequence $\left\{e^{n^2\pi^2 t}, n = 1, 2, ..., t \in [0, t_2]\right\}$is minimal [5]. In virtue of Theorem 1.5 of [5] for each $\varepsilon > 0$ there exists a positive constant K_ε such that the biorthogonal sequence $\{w_n(t), n = 1, 2, ..., t \in [0, t_2]\}$ satisfies the condition

$$\|w_n(\cdot)\| < K_\varepsilon e^{\varepsilon n^2\pi^2}, n = 1, 2, ...,. \tag{3.12}$$

The positive constant ε can be chosen such that $t_2 - \varepsilon > 0$.

By the Minkowsky inequality and (3.12) one can show that

$\sum_{n=1}^{p} \sum_{m=1}^{p} c_n e^{-n^2\pi^2 t_2} \left(\int_0^{t_2} w_n(t)\, w_m(t)\, dt\right) e^{-m^2\pi^2 t_2} c_m = \int_0^{t_2} \left(\sum_{n=1}^{p} c_n e^{-n^2\pi^2 t_2} w_n(t)\, dt\right)^2 dt \leq$

$\leq \int_0^{t_2} \sum_{n=1}^{p} |c_n|^2 \sum_{n=1}^{p} \left|e^{-n^2\pi^2 t_2} w_n(t)\right|^2 dt = \sum_{n=1}^{p} |c_n|^2 \sum_{n=1}^{p} e^{-2n^2\pi^2 t_2} \int_0^{t_2} |w_n(t)|^2 dt \leq$

$\leq \sum_{n=1}^{p} |c_n|^2 \sum_{n=1}^{p} e^{-2n^2\pi^2 t_2} \|w_n(\cdot)\|^2 \leq K_\varepsilon^2 \sum_{n=1}^{p} |c_n|^2 \sum_{n=1}^{p} e^{-2n^2\pi^2 (t_2 - \varepsilon)}.$

The series $\sum_{n=1}^{\infty} e^{-2n^2\pi^2(t_2-\varepsilon)}$ converges for any $t_2, \varepsilon, t_2 > \varepsilon$, so $\sum_{n=1}^{p} e^{-2n^2\pi^2(t_2-\varepsilon)} \leq M$, where M is a positive constant.

Hence

$$\sum_{n=1}^{p}\sum_{m=1}^{p} c_n e^{-n^2\pi^2 t_2}\left(\int_0^{t_2} w_n(t)\, w_m(t)\, dt\right) e^{-m^2\pi^2 t_2} c_m \le K_\varepsilon^2 M \sum_{n=1}^{p} |c_n|^2 \tag{3.13}$$

for every finite sequence $\{c_1, c_2, ..., c_p\}$. Obviously the sequence $\left\{ h_n(t) = \left\{ \begin{array}{cc} e^{-n^2\pi^2 t_2} w_n(t-t_2), & t \in [t_2, 2t_2], \\ 0, & t \in [0, t_2) \end{array} \right., n = 1, 2, ..., \right\}$ is the biorthogonal sequence to the sequence $\left\{ e^{n^2\pi^2 t}, n = 1, 2, ..., t \in [0, t_1] \right\}$, and $\left(\int_0^{t_1} h_n(t)\, h_m(t)\, dt\right) = e^{-n^2\pi^2 t_2}\left(\int_{t_2}^{2t_2} w_n(t-t_2)\, w_m(t-t_2)\, dt\right) e^{-m^2\pi^2 t_2} = e^{-n^2\pi^2 t_2}\left(\int_0^{t_2} w_n(t)\, w_m(t)\, dt\right) e^{-m^2\pi^2 t_2}$,
so it follows from (3.13) that

$$\sum_{n=1}^{p}\sum_{m=1}^{p} c_n \left(\int_0^{t_1} h_n(t)\, h_m(t)\, dt\right) c_m \le K_\varepsilon^2 M \sum_{n=1}^{p} |c_n|^2 .$$

Hence [9]

$$\sum_{n=1}^{p}\sum_{m=1}^{p} c_n \left(\int_0^{2t_1} e^{n^2\pi^2\tau} e^{m^2\pi^2\tau}\right) c_m d\tau \ge \gamma \sum_{n=1}^{p} |c_n|^2, p = 1, 2, ..., \tag{3.14}$$

for every finite sequence $\{c_1, c_2, ..., c_p\}$,where $\gamma = \frac{1}{K_\varepsilon^2 M} > 0$. It proves that the sequence $\left\{ e^{n^2\pi^2 t}, t \in [0, t_1],\ n = 1, 2, ... \right\}$ is strongly minimal for any $t_1 > 0$.

4 Approximation Theorems

As was said at the end of the previous section the condition $\lim\limits_{n\to\infty} \lambda_n^{\min} > 0$ in general can be checked by numerical methods. The problem appears to be rather difficult in general.

However there are sequences for which the validity of above inequality can be easily established. For example, every orthonormal sequence is strongly minimal.

Below we will show that if the sequence

$$\{y_j \in X, j = 1, 2, ...\}$$

can be approximated in the some sense by strongly minimal sequence

$$\{x_j \in X, j = 1, 2, ...\},$$

then it is also strongly minimal.

Theorem 4.1 *If the sequence $\{x_j \in X, j = 1, 2, ...\}$ is strongly minimal, let the sequence $\{y_j \in X, j = 1, 2, ...\}$ be such that the sequence $\{P_n y_j - x_j, j = 1, 2, ...\}$ is linear independent and*

$$\left\| \sum_{j=1}^{n} c_j \left(y_j - x_j\right) \right\| \leq q \left\| \sum_{j=1}^{n} c_j x_j, \right\|, n = 1, 2, ... \ , \tag{4.1}$$

where $\{c_j, j = 1, 2, ...\}$ is any sequence of complex numbers, q is a constant, $0 < q < 1$, then the sequence $\{y_j \in X, j = 1, 2, ...\}$ also is strongly minimal.

Proof. Let $\{c_k, k = 1, 2, ...\}$ be an arbitrary sequence of complex number. Denote:

$$x^0 = \sum_{k=1}^{n} c_k x_k, \ x^1 = \sum_{k=1}^{n} c_k \left(x_k - y_k\right). \tag{4.2}$$

From (4.2) it follows, that

$$x^0 = x^1 + \sum_{k=1}^{n} c_k y_k, \ n = \ 1, 2, ... \tag{4.3}$$

By (4.1) we obtain that

$$\left\|x^1\right\| \leq q \left\|x^0\right\|. \tag{4.4}$$

Hence using (4.4) in (4.3) we obtain

$$\left\|x^0\right\| \leq \frac{1}{1-q} \left\| \sum_{k=1}^{n} c_k y_k \right\|, \ n = \ 1, 2, ... \tag{4.5}$$

Since the sequence $\{x_j \in X, j = 1, 2, ...\}$ is strongly minimal and $x^0 = \sum_{k=1}^{n} c_k x_k$, we have

$$\sum_{k=1}^{n} |c_k|^2 \leq \frac{1}{\alpha^2} \left\|x^0\right\|^2, \ n = 1, 2, ..., \tag{4.6}$$

for some $\alpha > 0$.

By (4.6) and (4.5) we obtain $\alpha^2 \sum_{k=1}^{n} |c_k|^2 \leq \frac{1}{1-q} \left\|\sum_{k=1}^{n} c_k y_k\right\|, \ n = 1, 2, ...,$ so

$$\alpha^2 (1-q)^2 \left(\sum_{k=1}^{n} |c_k|^2 \right) \leq \left\| \sum_{k=1}^{n} c_k y_k \right\|, \ n = \ 1, 2, ..., . \tag{4.7}$$

Using in (4.7) the formula (2.14) we obtain

$$\gamma \left(\sum_{k=1}^{n} |c_k|^2 \right) \leq \sum_{k=1}^{n} \sum_{l=1}^{n} c_k (y_k, y_l) \overline{c_l}, \gamma = \alpha^2 (1-q)^2 > 0 \tag{4.8}$$

Let $\mu_{\min}^{[n]}$ be a minimal eigenvalue of the Gram matrix $G_n = \{(y_k, y_l), k, l = 1, 2....\}$ for the sequence $\{y_j, j = 1, 2, ..., n\}$. From (4.8), it follows that $\lim_{n\to\infty} \mu_{\min}^{[n]} \geq \gamma > 0$.

This proves the theorem.

4.1 Example

Let $X = l_2$ be the Hilbert space of square summable sequences. Consider the evolution system

$$\begin{cases} \dot{x}_k(t) = \lambda_k x_k(t) + u(t), & k = 1, 2, ..., \quad 0 < t < t_1, \\ x_k(0) = x_{k0}, n = 1, 2, ..., & k = 1, 2, ..., \end{cases} \tag{4.9}$$

where $u(t), 0 < t < t_1$ is a scalar control function,

$\{x_k(t), k = 1, 2, ..., \}, \{x_{k0}, k = 1, 2, ..., \} \in l^2$, the complex numbers λ_k, $k = 1, 2, ...$,belong to the strip $\{z \in \mathbb{C} : |\mathrm{Re}\, z| \leq \gamma\}$, i.e. $|\mathrm{Re}\, \lambda_k| \leq \gamma, k = 1, 2, ..., $.

Definition 4.1 *Equation (4.9) is said to be exact null-controllable on* $[0, t_1]$ *by controls vanishing after time moment* t_2, *if for each* $x_0(\cdot) = \{x_{k0}, k = 1, 2, ..., \} \in l_2$ *there exists a control* $u(\cdot) \in L_2[0, t_2], u(t) = 0$ *a.e. on* $[t_2, +\infty)$ *such that*

$$x_k(t) \equiv 0, \ k = 1, 2, ..., \forall t \geq t_1.$$

Control problem (4.9) can be written in the form of (1.1), where $x(t) = \{x_k(t), k = 1, 2, ..., \} \in l^2, u(\cdot) \in L_2[0, t_1]$; the self-adjoint operator $A : l_2 \to l_2$ is defined for $x = \{x_k, k = 1, 2, ..., \} \in l_2$ by

$$Ax = \{\lambda_k x_k, k = 1, 2, ..., \} \tag{4.10}$$

with domain $D(A) = \{x \in l_2 : Ax \in l_2\}$, and the unbounded operator B is defined by

$$Bu = bu, u \in \mathbb{R}, \tag{4.11}$$

where $b = \{1, 1, ..., 1, ...\} \notin l_2$.

One can show that all the assumptions imposed on equation (1.1) are fulfilled for equation (4.9) with $T = 0$.

Obviously, the numbers $\lambda_k,\ k = 1, 2, ...,$ are eigenvalues of the operator A defined above; the sequences $e_k = \left\{ \underbrace{0, ..., 0, 1, 0, ..., 0}_{1 \text{ on } k\text{-th place}} \right\}$ are corresponding eigenvectors, forming the Riesz basic of l_2, so $b_j = 1, j = 1, 2, ...,$.

Together with system (4.9) consider the other evolution system

$$\left\{ \begin{array}{ll} \dot{x}_k(t) = \mu_k x_k(t) + u(t), & n = 1, 2, ..., \quad 0 < t < t_1, \\ x_k(0) = x_{k0}, k = 1, 2, ..., & n = 1, 2, ..., \end{array} \right. \tag{4.12}$$

where

$$\mu_k = \lambda_k + O\left(\frac{1}{k}\right), k = 1, 2, ...,. \tag{4.13}$$

Proposition 1 *If system (4.9) is exact null-controllable on $[0, t_1]$ by controls vanishing after time moment t_2, then the same is valid for system (4.12).*

Proof. From the Caushy-Schvartz inequality it follows that
$\int_0^{t_2} \left| \sum_{k=1}^n c_k \left(e^{-\mu_k t} - e^{-\lambda_k t} \right) \right|^2 dt \le \sum_{k=1}^n |c_k|^2 \int_0^{t_2} \sum_{k=1}^n \left| e^{-\mu_k t} - e^{-\lambda_k t} \right|^2 dt =$
$= \sum_{k=1}^n |c_k|^2 \int_0^{t_2} \sum_{k=1}^n e^{-2\lambda_k t} \left| e^{O\left(\frac{1}{k}\right)t} - 1 \right|^2 dt \le \sum_{k=1}^n |c_k|^2 \int_0^{t_2} e^{2\gamma t} \sum_{k=1}^n \left| e^{O\left(\frac{1}{k}\right)t} - 1 \right|^2 dt.$
The series $\sum_{k=1}^\infty \left| e^{O\left(\frac{1}{k}\right)t} - 1 \right|^2$ converges for any $t \ge 0$. Denote

$$M(t_2) = \int_0^{t_2} e^{2\gamma t} \sum_{k=1}^\infty \left| e^{O\left(\frac{1}{k}\right)t} - 1 \right|^2 dt. \tag{4.14}$$

Hence

$$\int_0^{t_2} \left| \sum_{k=1}^n c_k \left(e^{-\mu_k t} - e^{-\lambda_k t} \right) \right|^2 dt \le M(t_2) \sum_{k=1}^n |c_k|^2. \tag{4.15}$$

By Theorem 3.2 we have the sequence $\left\{ e^{-\lambda_j t}, t \in [0, t_2],\ j = 1, 2, ... \right\}$ to be strongly minimal, so

$$\sum_{k=1}^n |c_k|^2 \le \frac{1}{\alpha^2} \int_0^{t_2} \left| \sum_{k=1}^n c_k e^{-\lambda_k t} \right|^2 dt \text{ for some } \alpha > 0. \tag{4.16}$$

Joining (4.15) and (4.16) we obtain

$$\int_0^{t_2} \left| \sum_{k=1}^{n} c_k \left(e^{-\mu_k t} - e^{-\lambda_k t} \right) \right|^2 dt \le q \int_0^{t_2} \left| \sum_{k=1}^{n} c_k e^{-\lambda_k t} \right|^2 dt, \qquad (4.17)$$

where $q = \frac{M(t_2)}{\alpha}$.

Since from (4.14) it follows that $\lim\limits_{t_1 \to \infty} M(t_2) = 0$, one can choose the number t_2 such that $0 < q < 1$. Hence conditions (4.17) are the same as (4.1) for $x_k = e^{-\lambda_k t}, y_k = e^{-\mu_k t}, k = 1, 2, ..., t \in [0, t_2]\,; q = \frac{M(t_2)}{\alpha^2}$.

As it was said abov by Theorem 3.2 we have the sequence$\left\{e^{-\lambda_j t}, t \in [0, t_2]\,,\ j = 1, 2, ...\right\}$ to be strongly minimal .

In accordance with Theorem 4.1 the sequence $\left\{y_k = e^{-\mu_k t}, k = 1, 2, ..., t \in [0, t_2]\right\}$ is also strongly minimal, provided that t_2 is chosen such that $\frac{M(t_2)}{\alpha^2} < 1$. In accordance with Theorem 3.1 the strong minimality of the sequence $\left\{y_k = e^{-\mu_k t}, k = 1, 2, ..., t \in [0, t_2]\right\}$ provides the zero controllability of equation (4.11) on $[0, t_1]$ by controls vanishing after time moment t_2, $\frac{M(t_2)}{\alpha^2} < 1$, for any $t_1 \ge t_2$.

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